# A geometrical characterization of a class of nowhere differentiable functions

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The aim of this talk is to give a geometrical characterization of a class of everywhere continuous and nowhere differentiable functions. The word "geometrical" is used, since this characterization is based on an inequality for a family of parabolas with vertices on the graph of a function (see $(\mathrm{F1})_c$ below). As an application of this characterization, we show that the inf-convolution of a function in this class possesses remarkable properties. We also investigate a structure of this class.

First of all, we give the definition of the class $\mathcal{P}$, which is the central subject of our study in this talk. Let us denote by $C_p(\mathbb{R})$ the set of all continuous and periodic functions from $\mathbb{R}$ to $\mathbb{R}$ with period 1. Throughout this talk, we assume that $r$ is an integer such that $r \geq 2$. Given a function $f \in C_p(\mathbb{R})$, we consider, for each $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $k \in \mathbb{Z}$ and $y \in (0,1)$, the first-order forward and backward differences of $f$ at $(k+y)/r^n$ defined, respectively, by

$$\delta^+_{n,k}(y;f) = \frac{f\left(\frac{k+1}{r^n}\right) - f\left(\frac{k+y}{r^n}\right)}{\frac{1-y}{r^n}}, \quad \delta^-_{n,k}(y;f) = \frac{f\left(\frac{k+y}{r^n}\right) - f\left(\frac{k}{r^n}\right)}{\frac{y}{r^n}}.$$

**Definition.** For any given constant $c > 0$, a function $f \in C_p(\mathbb{R})$ belongs to $\mathcal{P}_c$ exactly when

$$\delta^+_{n,k}(y;f) - \delta^-_{n,k}(y;f) \leq -c$$

for all $(n,k,y) \in \mathbb{N}_0 \times \mathbb{Z} \times (0,1)$. We use the notation $\mathcal{P} = \bigcup_{c>0} \mathcal{P}_c$. Note that both $\mathcal{P}_c$ and $\mathcal{P}$ depend on the choice of $r$ though we omit it in our notation.

Inequality above can be written equivalently as

$$\Delta_{n,k}(y;f) \leq -2cr^n,$$

where $\Delta_{n,k}(y;f)$ is the second-order central difference defined by

$$\Delta_{n,k}(y;f) = 2r^n(\delta^+_{n,k}(y;f) - \delta^-_{n,k}(y;f)).$$

Thus, a function $f \in \mathcal{P}$ satisfies a concavity–type estimate.

We will show that each function in $\mathcal{P}$ is nowhere differentiable and that we can systematically construct a large number of examples of functions in $\mathcal{P}$ through an explicit formula.

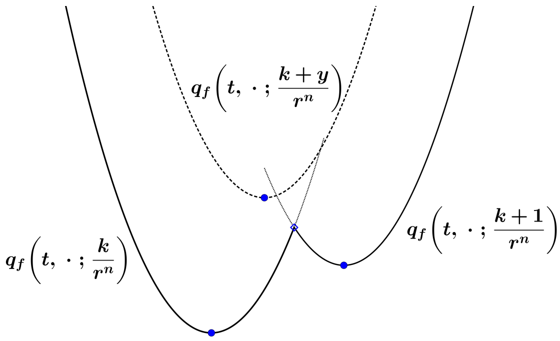
Now, we explain our geometrical characterization. For a function $f \in C_p(\mathbb{R})$, let us consider the family of parabolas $\{q_f(t,x;z)\}_{z\in\mathbb{R}}$ defined by

$$q_f(t,x;z) = f(z) + \frac{1}{2t}(x-z)^2, \quad (t,x,z) \in (0,\infty) \times \mathbb{R} \times \mathbb{R}.$$

We will show that a function $f \in C_p(\mathbb{R})$ belongs to $\mathcal{P}_c$ if and only if

$(\mathrm{F1})_c$ For all $(n,k,y) \in \mathbb{N}_0 \times \mathbb{Z} \times (0,1)$ and $t \geq 1/(2cr^n)$,

$$q_f\left(t, x; \frac{k+y}{r^n}\right) \geq \min\left\{ q_f\left(t, x; \frac{k}{r^n}\right), q_f\left(t, x; \frac{k+1}{r^n}\right) \right\}, \quad x \in \mathbb{R}.$$




We apply $(\mathrm{F1})_c$ to the inf-convolution $\{H_t f\}_{t>0}$ defined by

$$H_t f(x) = \inf_{z \in \mathbb{R}} q_f(t, x; z), \quad (t,x) \in (0,\infty) \times \mathbb{R}$$

for any given function $f \in \mathcal{P}_c$.

We also investigate a structure of $\mathcal{P}$. For this purpose, let us define the operator $U : \psi \in C_p(\mathbb{R}) \to U_\psi \in C_p(\mathbb{R})$ by

$$U_\psi(y) = \sum_{j=0}^{\infty} \frac{1}{r^j} \psi(r^j y), \quad y \in \mathbb{R}.$$

It is easy to see that $U$ is bijective. We would like to know the structure of $U^{-1}(\mathcal{P})$. Although we cannot determine it completely, we will show that

$$SC_0(\mathbb{R}) \setminus \{0\} \subset U^{-1}(\mathcal{P}), \quad \left( \bigcup_{\alpha>0} SC_\alpha(\mathbb{R}) \setminus SC_0(\mathbb{R}) \right) \cap U^{-1}(\mathcal{P}) \neq \emptyset,$$
$$\left( \bigcup_{\alpha>0} SC_\alpha(\mathbb{R}) \right) \cap U^{-1}(\mathcal{N} \setminus \mathcal{P}) \neq \emptyset,$$

where

$$SC_\alpha(\mathbb{R}) = \left\{ \psi \in C_p(\mathbb{R}) \,\middle|\, \psi(x) + \tfrac{\alpha}{2} x(1-x) \text{ is concave on } [0,1] \right\}, \quad \alpha \geq 0,$$
$$\mathcal{N} = \left\{ \xi \in C_p(\mathbb{R}) \,\middle|\, \xi \text{ is nowhere differentiable on } \mathbb{R} \right\}.$$

From these results, we see that the Takagi function belongs to $\mathcal{P}$ but a Weierstrass–type function does not belong to $\mathcal{P}$.

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